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MODIFIED BUSEMANN–PETTY PROBLEM ON SECTIONS OF CONVEX BODIES

BY

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ABSTRACT

The Busemann–Petty problem asks whether convex origin-symmetric bodies in $\mathbb{R}^n$ with smaller central hyperplane sections necessarily have smaller n-dimensional volume. It is known that the answer is affirmative if $n \leq 4$ and negative if $n \geq 5$. In this article we replace the assumptions of the original Busemann–Petty problem by certain conditions on the volumes of central hyperplane sections so that the answer becomes affirmative in all dimensions.

1. Introduction

The classical Minkowski's uniqueness theorem states that an origin-symmetric star body in $\mathbb{R}^n$ is uniquely determined by the volumes of its central hyperplane sections in all directions; see, for example, [K5, Corollary 3.9]. This result provides a strong intuition towards an affirmative answer in the following Busemann–Petty problem [BP]: given two convex origin-symmetric bodies K and L in $\mathbb{R}^n$ such that

$$\mathrm{vol}_{n-1}(K \cap H) \leq \mathrm{vol}_{n-1}(L \cap H)$$

for every central hyperplane H in $\mathbb{R}^n$, does it follow that

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)?$$

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The solution was completed a few years ago and appeared as the result of work of many mathematicians (see [GKS], [Zh] or [K5, Chapter 5] for the solution and historical details). Surprisingly, the answer is affirmative only if the dimension $n \le 4$, and it is negative if $n \ge 5$. In view of this answer, it is natural to ask what information about the volumes of central hyperplane sections of two bodies does allow one to compare the volumes of these bodies in all dimensions. Our main result suggests an answer to this question.

For an origin-symmetric convex body K in $\mathbb{R}^n$, consider the section function

$$S_K(\xi) = \mathrm{vol}_{n-1}(K \cap \xi^{\perp}), \quad \xi \in S^{n-1},$$

where $\xi^{\perp}$ is the central hyperplane in $\mathbb{R}^n$ orthogonal to ξ. We extend S_K from the sphere to the whole $\mathbb{R}^n$ as a homogeneous function of degree -1. Our goal is to find a condition in terms of the section functions of two bodies only that allows one to compare the n-dimensional volumes of these bodies. We prove in this paper that, for two origin-symmetric smooth bodies K, L in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$, $\alpha \ge n-4$, the inequalities

$$(1) \qquad (-\Delta)^{\alpha/2} S_K(\xi) \le (-\Delta)^{\alpha/2} S_L(\xi), \quad \forall \xi \in S^{n-1}$$

imply that $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$, while for $\alpha < n-4$ this is not necessarily true. Here Δ is the Laplace operator on $\mathbb{R}^n$, and the fractional powers of the Laplacian are defined by

$$(-\Delta)^{\alpha/2} f = \frac{1}{(2\pi)^n} (|x|_2^{\alpha} \hat{f}(x))^{\wedge},$$

where the Fourier transform is considered in the sense of distributions, and $|x|_2$ stands for the Euclidean norm in $\mathbb{R}^n$. Of course, if α is an even integer and f is an even distribution, we get the Laplacian applied $\alpha/2$ times. The fact that both sides of (1) represent continuous functions of the variable ξ follows from [K5, Lemma 3.16].

This result means that one has to differentiate the section functions at least $n-4$ times in order to compare the n-dimensional volumes. The case $\alpha = 0$ corresponds to the original Busemann–Petty problem, so our result can also be considered as a "continuous" generalization of the problem. Other generalizations of the Busemann–Petty problem and related open questions can be found in [BZ], [K2], [K3], [K4], [MP], [RZ], [Y], [Zv].

Let us briefly outline the idea of the proof. As shown in [K1], the section function can be expressed in terms of the Fourier transform, as follows:

$$(2) \qquad S_K(\xi) = \frac{1}{\pi(n-1)} (\|x\|_K^{-n+1})^{\wedge}(\xi),$$

so the condition (1) can be written as

$$(3) \qquad (|x|_2^\alpha \|x\|_K^{-n+1})^\wedge \leq (|x|_2^\alpha \|x\|_K^{-n+1})^\wedge.$$

Now let us write the volume in polar coordinates and use a spherical version of Parseval's formula from [K2], which allows one to remove the Fourier transforms of homogeneous functions in the integrals over the sphere under the condition that the degrees of homogeneity of these functions add up to $-n$:

$$\begin{aligned} n \operatorname{vol}_n(K) &= \int_{S^{n-1}} \|x\|_K^{-n} dx = \int_{S^{n-1}} |x|_2^{-\alpha} \|x\|_K^{-1} |x|_2^\alpha \|x\|_K^{-n+1} dx \\ &= \frac{1}{(2\pi)^n} \int_{S^{n-1}} (|x|_2^{-\alpha} \|x\|_K^{-1})^\wedge(\xi) (|x|_2^\alpha \|x\|_K^{-n+1})^\wedge(\xi) d\xi. \end{aligned}$$

Suppose that the distribution $|x|_2^{-\alpha}\|x\|_K^{-1}$ is positive definite, so its Fourier transform is non-negative. Then the latter equality combined with (3) implies that

$$n \operatorname{vol}_n(K) \leq \int_{S^{n-1}} \|x\|_K^{-1} \|x\|_L^{-n+1} dx,$$

and applying Hölder's inequality to the right-hand side we get that $\operatorname{vol}_n(K) \leq \operatorname{vol}_n(L)$. On the other hand, if $|x|_2^{-\alpha}\|x\|_K^{-1}$ is not positive definite one can construct a counterexample using a more or less standard perturbation procedure.

Thus, the problem is essentially reduced to the question, for which α is the distribution $|x|_2^{-\alpha}\|x\|_K^{-1}$ positive definite, for every origin-symmetric convex body K in $\mathbb{R}^n$. Note that for $\alpha = 0$ this happens only if the dimension $n \leq 4$, as proved in [GKS]. We prove that this function is positive definite for $\alpha \geq n-4$ and any symmetric convex body K in $\mathbb{R}^n$ by an argument modifying the proof from [GKS]. If $\alpha < n-4$ we construct examples of bodies for which this distribution is not positive definite. The latter requires a substantial technical effort.

2. Positive definite distributions of the form $|x|_2^{-r}\|x\|_K^{-s}$

Let K be a convex origin-symmetric body in $\mathbb{R}^n$. Our definition of a convex body assumes that the origin is an interior point of K. The **radial function** of K is given by

$$\rho_K(x) = \max\{a > 0 : ax \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

The Minkowski norm of K is defined as

$$\|x\|_K = \min\{a \geq 0 : x \in aK\};$$

clearly $\rho_K(x) = \|x\|_K^{-1}$.

Writing the volume of K in polar coordinates, one can express the volume in terms of the Minkowski norm:

$$\mathrm{vol}_n(K) = \frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta. \tag{4}$$

We say that a body K is infinitely smooth if its radial function ρ_K restricted to the unit sphere S^{n-1} belongs to the space $C^\infty(S^{n-1})$ of infinitely differentiable functions on the unit sphere. Note that a simple approximation argument reduces the original Busemann–Petty problem (as well as all generalizations mentioned in the introduction) to the case where the bodies K and L are infinitely smooth.

Throughout the paper we use the Fourier transform of distributions. The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$ for every test function ϕ from the Schwartz space $\mathcal{S}$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. For any even distribution f, we have $(\hat{f})^\wedge = (2\pi)^n f$.

A distribution is **positive definite** if its Fourier transform is a positive distribution in the sense that $\langle \hat{f}, \phi\rangle \geq 0$ for every non-negative test function ϕ; see, for example, [GV, p. 152].

Let f be an integrable continuous function on $\mathbb{R}$, m-times continuously differentiable in some neighborhood of zero, $m \in \mathbb{N}$. For a number $q \in (m-1, m)$ the **fractional derivative** of the order q of the function f at zero is defined by

$$f^{(q)}(0) = \frac{1}{\Gamma(-q)}\int_0^\infty t^{-1-q}\Big(f(t) - f(0) - tf'(0) - \cdots - \frac{t^{m-1}}{(m-1)!}f^{(m-1)}(0)\Big)dt.$$

Note that without dividing by $\Gamma(-q)$ the expression for the fractional derivative represents an analytic function in the domain $\{q \in \mathbb{C}, -1 < \mathrm{Re}\, q < m\}$ not including integers and has simple poles at non-negative integers. The function $\Gamma(-q)$ is analytic in the same domain and also has simple poles at non-negative integers. Therefore, after division we get an analytic function in the whole domain $\{q \in \mathbb{C}, -1 < \mathrm{Re}\, q < m\}$, which also defines fractional derivatives of integer orders. Moreover, computing the limit as $q \to k$, where k is a non-negative integer and $k < m$, we see that the fractional derivatives of integer orders coincide with usual derivatives up to a sign:

$$f^{(k)}(0) = (-1)^k \frac{d^k}{dt^k} f(t)|_{t=0}.$$

More details on fractional derivatives may be found in [K5, Section 2.6].

For $\xi \in S^{n-1}$, consider a function $A_{K,\xi,p}$ on $\mathbb{R}$,

$$A_{K,\xi,p}(t) = \int_{K\cap\langle x,\xi\rangle=t} |x|_2^{-p} dx,$$

where $p < n-1$.

In this section we establish some regularity properties of the function $A_{K,\xi,p}$ and express its fractional derivatives in terms of the Fourier transform. We assume that K is an infinitely smooth body.

For a real number q define the ceiling function $\lceil q \rceil$, which gives the smallest integer greater than or equal to q.

LEMMA 2.1: *Let $\xi \in S^{n-1}$, $k \in \mathbb{N}$, $0 \le p < n-k-1$. Then the function $A_{K,\xi,p}$ is k-times continuously differentiable (uniformly with respect to ξ) in some neighborhood of zero.*

For fixed $q \in \mathbb{C}$, the fractional derivative $A^{(q)}_{K,\xi,p}(0)$ is a continuous function of the variable $\xi \in S^{n-1}$, and, for fixed $\xi \in S^{n-1}$, it is an analytic function of q in the domain $\{q \in \mathbb{C}\colon -1 < \lceil \mathrm{Re}\, q \rceil < n-p-1\}$, with convergence in the derivatives by q being uniform with respect to ξ.

The proof is similar to that of [K5, Lemma 2.4]. The only difference is that in our case the function is differentiable only up to a certain order. To explain this, write the function in the form

$$A_{K,\xi,p}(t) = \int_{S_t^{n-2}} \left(\int_0^{\rho_{K\cap H_t}(\theta)} r^{n-2}(r^2+t^2)^{-p/2} dr \right) d\theta,$$

where $\rho_{K\cap H_t}(\theta)$ is the radial function of the body $K \cap H_t$ and S_t^{n-2} is the unit sphere in $H_t = \{x \in \mathbb{R}^n\colon \langle x,\xi\rangle = t\}$. If we differentiate by t too many times the integral stops being convergent when $t=0$, which is why we have restrictions on k and q.

The following Lemma is a generalization of Theorem 2 from [GKS].

LEMMA 2.2: *Let K be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, $q > -1$, $q \ne n-p-1$ and $0 \le p < n - \lceil q \rceil - 1$. Then for every $\xi \in S^{n-1}$,*

$$A^{(q)}_{K,\xi,p}(0) = \frac{\cos(\pi q/2)}{\pi(n-p-q-1)} (\|x\|_K^{-n+p+q+1} \cdot |x|_2^{-p})^\wedge(\xi).$$

Proof: We simply write $\|\cdot\|$ for $\|\cdot\|_K$. By [K5, Lemma 3.16], $(\|x\|^{-n+p+q+1} \cdot |x|_2^{-p})^\wedge$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$.

Suppose first that $-1 < q < 0$. The function

$$A_{K,\xi,p}(t) = \int_{K\cap\langle x,\xi\rangle=t} |x|_2^{-p} dx = \int_{\langle x,\xi\rangle=t} \chi(\|x\|)|x|_2^{-p} dx$$

is even. Applying Fubini's theorem and passing to spherical coordinates, we get

$$\begin{aligned}
A^{(q)}_{K,\xi,p}(0) &= \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}A_{K,\xi,p}(t)dt\\
&= \frac{1}{2\Gamma(-q)}\int_{-\infty}^\infty |t|^{-q-1}A_{K,\xi,p}(t)dt\\
&= \frac{1}{2\Gamma(-q)}\int_{-\infty}^\infty |t|^{-q-1}\int_{\langle x,\xi\rangle=t}\chi(\|x\|)|x|_2^{-p}dxdt\\
&= \frac{1}{2\Gamma(-q)}\int_{\mathbb{R}^n}|\langle x,\xi\rangle|^{-q-1}\chi(\|x\|)|x|_2^{-p}dx\\
&= \frac{1}{2\Gamma(-q)}\int_{S^{n-1}}|\langle\theta,\xi\rangle|^{-q-1}\int_0^\infty r^{-q-1}\chi(r\|\theta\|)r^{-p}r^{n-1}drd\theta\\
&= \frac{1}{2\Gamma(-q)}\int_{S^{n-1}}|\langle\theta,\xi\rangle|^{-q-1}\int_0^{1/\|\theta\|} r^{n-p-q-2}drd\theta\\
&= \frac{1}{2\Gamma(-q)(n-p-q-1)}\int_{S^{n-1}}|\langle\theta,\xi\rangle|^{-q-1}\|\theta\|^{-n+p+q+1}d\theta.
\end{aligned}$$

Now we extend $A^{(q)}_{K,\xi,p}(0)$ to $\mathbb{R}^n$ as a homogeneous function of ξ of degree $-1-q$. Then for every even test function $\phi\in\mathcal{S}$,

$$\begin{aligned}
\langle A^{(q)}_{K,\xi,p}(0),\phi(\xi)\rangle =& \frac{1}{2\Gamma(-q)(n-p-q-1)}\\
&\times\int_{S^{n-1}}\|\theta\|^{-n+p+q+1}\int_{\mathbb{R}^n}|\langle\theta,\xi\rangle|^{-q-1}\phi(\xi)d\xi d\theta.
\end{aligned}$$

Using Lemma 5 from [GKS]

$$\begin{aligned}
&= \frac{-1}{4\Gamma(-q)\Gamma(1+q)(n-p-q-1)\sin(q\pi/2)}\\
&\qquad\times\int_{S^{n-1}}\|\theta\|^{-n+p+q+1}\int_{-\infty}^\infty |t|^q\hat{\phi}(t\theta)dtd\theta\\
&= \frac{-\sin(-\pi q)}{2\pi(n-p-q-1)\sin(q\pi/2)}\langle(\|x\|^{-n+p+q+1}\cdot|x|_2^{-p})^\wedge(\xi),\phi(\xi)\rangle.
\end{aligned}$$

The latter follows from the fact that $\Gamma(-q)\Gamma(q+1)=-\pi/\sin(q\pi)$ and the calculation

$$\begin{aligned}
\langle(\|x\|^{-n+p+q+1}\cdot|x|_2^{-p})^\wedge(\xi),\phi(\xi)\rangle &= \int_{R^n}\|x\|^{-n+p+q+1}\cdot|x|_2^{-p}\hat{\phi}(x)dx\\
&= \int_{S^{n-1}}\|\theta\|^{-n+p+q+1}\\
&\qquad\times\int_0^\infty t^{-n+p+q+1}t^{-p}t^{n-1}\hat{\phi}(t\theta)dtd\theta
\end{aligned}$$

$$= \int_{S^{n-1}} \|\theta\|^{-n+p+q+1} \int_0^\infty t^q \hat{\phi}(t\theta) dt d\theta.$$

We have proved that

$$\langle A^{(q)}_{K,\xi,p}(0), \phi(\xi)\rangle = \frac{\cos(\pi q/2)}{\pi(n+p-q-1)} \langle (\|x\|^{-n+p+q+1} \cdot |x|_2^{-p})^\wedge(\xi), \phi(\xi)\rangle$$

for $-1 < q < 0$. Since both $A^{(q)}_{K,\xi,p}(0)$ and $(\|x\|^{-n+p+q+1} \cdot |x|_2^{-p})^\wedge(\xi)$ are continuous functions of $\xi \in \mathbb{R}^n \setminus \{0\}$, we get the statement of the Lemma for $-1 < q < 0$.

To prove the Lemma for other values of q, we use the fact that for every even test function ϕ the functions

$$q \mapsto \langle A^{(q)}_{K,\xi,p}(0), \phi(\xi)\rangle$$

and

$$q \mapsto \frac{\cos(\pi q/2)}{\pi(n-p-q-1)} \langle (\|x\|^{-n+p+q+1} \cdot |x|_2^{-p})^\wedge(\xi), \phi(\xi)\rangle$$

are analytic in the domain $\{q \in \mathbb{C}\colon -1 < \lceil \mathrm{Re}\, q \rceil < n-p-1\}$. (The fact, that $(\|x\|^{-n+p+q+1} \cdot |x|_2^{-p})^\wedge(\xi)$ is analytic with respect to q, can be seen from the argument of [K5, Lemma 2.22].) The result of the Lemma follows, since these analytic functions coincide for $q \in (-1,0)$, ϕ is arbitrary and, by Lemma 2.1, the fractional derivative is a continuous function of ξ outside of the origin. ∎

LEMMA 2.3: *Let K be an origin-symmetric convex body in $\mathbb{R}^n$. Assume that $q \in (-1,2]$ and $0 \le p < n - \lceil q \rceil - 1$. Then $\|x\|_K^{-n+p+q+1} \cdot |x|_2^{-p}$ is a positive definite distribution on $\mathbb{R}^n$.*

Proof: First we prove that

$$A_{K,\xi,p}(t) \le A_{K,\xi,p}(0), \quad \text{for all } t \ge 0. \tag{5}$$

If $p = 0$, it follows from Brunn's theorem (see [K5, Theorem 2.3]) that the central hyperplane section of an origin-symmetric convex body has maximal volume among all hyperplane sections orthogonal to a given direction. If $p > 0$ one can see that

$$|x|_2^{-p} = p \int_0^\infty \chi(z|x|_2) z^{p-1} dz,$$

therefore

$$\begin{aligned} A_{K,\xi,p}(t) &= \int_{K\cap\langle x,\xi\rangle=t} |x|_2^{-p}dx \\ &= p\int_{K\cap\langle x,\xi\rangle=t}\int_0^\infty \chi(z|x|_2)z^{p-1}dzdx \\ &= p\int_0^\infty z^{p-1}\int_{K\cap\langle x,\xi\rangle=t}\chi(z|x|_2)dxdz \\ &= p\int_0^\infty z^{p-1}\int_{B_{1/z}\cap K\cap\langle x,\xi\rangle=t} dxdz \\ &\le p\int_0^\infty z^{p-1}\int_{B_{1/z}\cap K\cap\langle x,\xi\rangle=0} dxdz = A_{K,\xi,p}(0) \end{aligned}$$

by Brunn's theorem applied to the convex origin-symmetric body $B_{1/z}\cap K$, where $B_{1/z}$ is a ball of radius $1/z$.

Now consider $q\in(1,2)$. Here $\cos(q\pi/2)$ is negative, therefore we need to prove that $A^{(q)}_{K,\xi,p}(0)\le 0$. Using inequality (5), the formula for fractional derivatives for $q\in(1,2)$ and the fact that $A'(0)=0$ we get

$$\begin{aligned} A^{(q)}_{K,\xi,p}(0) &= \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}(A(t)-A(0)-tA'(0))dt \\ &= \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}(A(t)-A(0))dt \le 0, \end{aligned}$$

since $\Gamma(-q)$ is positive.

If $q\in(0,1)$, then $\cos(q\pi/2)$ is positive and

$$A^{(q)}_{K,\xi,p}(0) = \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}(A(t)-A(0))dt \ge 0,$$

since $\Gamma(-q)<0$ for these values of q.

Finally, if $q\in(-1,0)$ then $\cos(q\pi/2)$ is positive, $\Gamma(-q)$ is also positive and

$$A^{(q)}_{K,\xi,p}(0) = \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}A(t)dt \ge 0.$$

We still have to prove the Lemma for $q=0,1,2$.

When $q=0$, $\cos(\pi q/2)=1$ and

$$A^{(0)}_{K,\xi,p}(0) = (-1)^0 A_{K,\xi,p}(0) \ge 0.$$

When $q=2$, $\cos(\pi q/2)=-1$ and

$$A^{(2)}_{K,\xi,p}(0) = (-1)^2 A''_{K,\xi,p}(0) \le 0,$$

since $A_{K,\xi,p}(t)$ has maximum at 0.

When $q = 1$, take small $\varepsilon > 0$. By what we have just proved for non-integer q, for any non-negative test function ϕ,

$$\langle (|x|_2^{-p} \|x\|_K^{-n+p+2+\varepsilon})^\wedge, \phi \rangle \geq 0.$$

Since $\|x\|_K \leq C|x|_2$ for some C, it follows that

$$\|x\|_K^{-n+p+2+\varepsilon} |x|_2^{-p} \leq \tilde{C}|x|_2^{-n+2+\varepsilon} \leq \tilde{C}|x|_2^{-n+1},$$

the latter being a locally-integrable function on $\mathbb{R}^n$.

Set $g(x) = \tilde{C}|x|_2^{-n+1}|\hat{\phi}(x)|$ for $|x|_2 < 1$ and $g(x) = \tilde{C}|\hat{\phi}(x)|$ for $|x|_2 > 1$. The function $g(x)$ is integrable on $\mathbb{R}^n$ and for small ε we have that

$$\|x\|_K^{-n-p+2+\varepsilon}|x|_2^p \hat{\phi}(x) \leq g(x).$$

Therefore, by the Lebesgue dominated convergence theorem,

$$\langle (\|x\|_K^{-n+p+2}|x|_2^{-p})^\wedge, \phi \rangle = \int_{\mathbb{R}^n} \|x\|_K^{-n+p+2}|x|_2^{-p}\hat{\phi}(x)dx$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \|x\|_K^{-n+p+2+\varepsilon}|x|_2^{-p}\hat{\phi}(x)dx = \lim_{\varepsilon \to 0} \langle (\|x\|_K^{-n+p+2+\varepsilon}|x|_2^{-p})^\wedge, \phi \rangle \geq 0. \quad \blacksquare$$

3. The proof of the main result

THEOREM 3.1: *Let $\alpha \in [n-4, n-1)$, K and L be origin-symmetric infinitely smooth convex bodies in $\mathbb{R}^n$, $n \geq 4$, so that for every $\xi \in S^{n-1}$,*

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi). \tag{6}$$

Then

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L).$$

On the other hand, for any $\alpha \in [n-5, n-4)$ there are convex origin-symmetric bodies $K, L \in \mathbb{R}^n$, $n \geq 5$ that satisfy (6) *for every $\xi \in S^{n-1}$ but $\mathrm{vol}_n(L) < \mathrm{vol}_n(K)$.*

Proof of the affirmative part: Let $S_K(\xi) = \mathrm{vol}_{n-1}(K \cap \xi^\perp)$, $\xi \in S^{n-1}$, the central section function defined in the Introduction. Then, as proved in [K1],

$$S_K(\xi) = \frac{1}{\pi(n-1)} (\|x\|_K^{-n+1})^\wedge(\xi). \tag{7}$$

Extending $S_K(\xi)$ to $\mathbb{R}^n$ as a homogeneous function of degree -1 and using the definition of fractional powers of the Laplacian we get

$$(-\Delta)^{\alpha/2}S_L(\theta) = \frac{1}{\pi(n-1)}(|x|_2^{\alpha}\|x\|_L^{-n+1})^\wedge(\theta),$$

therefore

$$\begin{aligned}(2\pi)^n \int_{S^{n-1}} &\|x\|_K^{-1}\|x\|_L^{-n+1}dx \\ &= (2\pi)^n \int_{S^{n-1}} (|x|_2^{-\alpha}\|x\|_K^{-1})(|x|_2^{\alpha}\|x\|_L^{-n+1})dx \\ &= \int_{S^{n-1}} (|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge(\theta)(|x|_2^{\alpha}\|x\|_L^{-n+1})^\wedge(\theta)d\theta \\ &= \pi(n-1)\int_{S^{n-1}} (|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge(\theta)(-\Delta)^{\alpha/2}S_L(\theta)d\theta.\end{aligned}$$

Here we have used Parseval's formula on the sphere (see [K2, Lemma 3]) and (7).

By Lemma 2.3 with $p = \alpha$ and $q = n-\alpha-2$, $(|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge$ is a non-negative function on S^{n-1}, therefore using the condition of the theorem and repeating the above calculation in the opposite order, we get

$$\int_{S^{n-1}} \|x\|_K^{-1}\|x\|_K^{-n+1}dx \le \int_{S^{n-1}} \|x\|_K^{-1}\|x\|_L^{-n+1}dx.$$

Then by Hölder's inequality and the polar formula for the volume (4),

$$\begin{aligned} n\,\mathrm{vol}_n(K) &\le \left(\int_{S^{n-1}} \|\theta\|_K^{-n}d\theta\right)^{1/n}\left(\int_{S^{n-1}} \|\theta\|_L^{-n}d\theta\right)^{(n-1)/n} \\ &= n(\mathrm{vol}_n(K))^{1/n}(\mathrm{vol}_n(L))^{(n-1)/n},\end{aligned}$$

which yields the statement of the positive part of the theorem.

Proof of the negative part: Let $\alpha \in [n-5, n-4)$. We need to construct two convex origin-symmetric bodies $K, L \in \mathbb{R}^n$, $n \ge 5$ such that for every $\xi \in S^{n-1}$

$$(-\Delta)^{\alpha/2}S_K(\xi) \le (-\Delta)^{\alpha/2}S_L(\xi),$$

but

$$\mathrm{vol}_n(L) < \mathrm{vol}_n(K).$$

First let us prove the following Lemma.

LEMMA 3.2: *Let $\alpha \in [n-5, n-4)$. There exists an infinitely smooth origin-symmetric convex body L with positive curvature, so that*

$$\|x\|_L^{-1} \cdot |x|_2^{-\alpha}$$

is not a positive definite distribution.

Proof: First assume that $\alpha \in (n-5, n-4)$. Put $q = n - \alpha - 2$, so $q \in (2,3)$. Our goal is to construct a body L so that there is a $\xi \in S^{n-1}$ satisfying

$$\int_0^\infty t^{-q-1}\Big(A_{L,\xi,\alpha}(t) - A_{L,\xi,\alpha}(0) - A''_{L,\xi,\alpha}(0)\frac{t^2}{2}\Big)dt < 0. \tag{8}$$

If we construct such a body L, the result of this lemma immediately follows from Lemma 2.2 and the definition of fractional derivatives.

Consider the function

$$f(t) = (1 - t^2 - Nt^4)^{1/(n-\alpha-1)}.$$

Let a_N be the positive real root of the equation $f(t) = 0$. Define the body $L \in \mathbb{R}^n$ as follows:

$$L = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n \colon x_n \in [-a_N, a_N] \text{ and } \left(\sum_{i=1}^{n-1} x_i^2\right)^{1/2} \le f(x_n)\right\},$$

which is a strictly convex infinitely differentiable body.

Take ξ to be the unit vector in the direction of the x_n-axis. Then for $t \in [0, a_N]$,

$$A_{L,\xi,\alpha}(t) = \int_{S^{n-1}} \int_0^{f(t)} (t^2 + r^2)^{-\alpha/2} r^{n-2} dr\, d\theta$$
$$= C_n \int_0^{f(t)} (t^2 + r^2)^{-\alpha/2} r^{n-2} dr,$$

where $C_n = |S^{n-1}|$, and for $t > a_N$ we have $A_{L,\xi,\alpha}(t) = 0$.

One can compute

$$A_{L,\xi,p}(0) = \frac{C_n}{n - \alpha - 1}$$

and

$$A''_{L,\xi,p}(0) = -C_n\Big[\frac{\alpha}{n-\alpha-3} + \frac{2}{n-\alpha-1}\Big].$$

In order to estimate the integral (8), we split it into three parts: over $[0, b_N]$, $[b_N, a_N]$ and $[a_N, \infty)$, where b_N is the positive real root of the equation $1 - t^2 - Nt^4 = t^{q+1}$. Recall that a_N was defined as the positive real root of the equation $1-t^2-Nt^4=0$. It is easy to check that $a_N \simeq b_N \simeq N^{-1/4}$ for large N. Also, note that on $[0,a_N]$ we have $f(t)\ge 0$, and $f(t)\ge t$ if and only if $t\in[0,b_N]$.

First consider the interval $[0,b_N]$. For all t from this interval we have $t\le f(t)$. Then we can break the integral

$$\int_0^{f(t)} (t^2+r^2)^{-\alpha/2} r^{n-2}dr = I_1 + I_2$$

into two parts, where the first one can be estimated as follows:

$$I_1 = \int_0^t (t^2+r^2)^{-\alpha/2} r^{n-2}dr \le \int_0^t (r^2)^{-\alpha/2} r^{n-2}dr = \frac{t^{n-\alpha-1}}{n-\alpha-1},$$

and for the second one we will use the inequality

$$(1+x)^{-\gamma} \le 1-\gamma x + \frac{\gamma(\gamma+1)}{2}x^2, \quad \text{for } \gamma>0 \text{ and } 0<x<1.$$

Then

$$\begin{aligned}
I_2 &= \int_t^{f(t)} (t^2+r^2)^{-\alpha/2} r^{n-2}dr \\
&= \int_t^{f(t)} \Big(1+\frac{t^2}{r^2}\Big)^{-\alpha/2} r^{n-\alpha-2}dr \\
&\le \int_t^{f(t)} \Big(1-\frac{\alpha}{2}\frac{t^2}{r^2}+\frac{\frac{\alpha}{2}(\frac{\alpha}{2}+1)}{2}\frac{t^4}{r^4}\Big) r^{n-\alpha-2}dr \\
&= \Big[\frac{r^{n-\alpha-1}}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2 r^{n-\alpha-3}}{n-\alpha-3} + \frac{\frac{\alpha}{2}(\frac{\alpha}{2}+1)}{2}\frac{t^4 r^{n-\alpha-5}}{n-\alpha-5}\Big]_t^{f(t)} \\
&= \frac{f^{n-\alpha-1}(t)}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3} f^{n-\alpha-3}(t) \\
&\quad + \frac{\frac{\alpha}{2}(\frac{\alpha}{2}+1)}{2}\frac{t^4}{n-\alpha-5} f^{n-\alpha-5}(t) + Ct^{n-\alpha-1} \\
&\le \frac{f^{n-\alpha-1}(t)}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3} f^{n-\alpha-3}(t) + Ct^{n-\alpha-1} \\
&= \frac{1-t^2-Nt^4}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3}(1-t^2-Nt^4)^{\frac{n-\alpha-3}{n-\alpha-1}} + Ct^{n-\alpha-1}
\end{aligned}$$

for some constant C. The last inequality follows from $f(t)\ge 0$ on $[0,b_N]$ and $\alpha\in(n-5,n-4)$.

Using the inequality

$$(1-x)^{\gamma} \ge 1-\gamma x(1-x)^{\gamma-1}, \quad \text{for } 0<\gamma<1 \text{ and } 0<x<1,$$

applied to the second term in the previous expression, we get

$$\begin{aligned}I_2 \le & \frac{1-t^2-Nt^4}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3}\times \\ & \times\Big(1-\frac{n-\alpha-3}{n-\alpha-1}(1-t^2-Nt^4)^{\frac{n-\alpha-3}{n-\alpha-1}-1}(t^2+Nt^4)\Big)+Ct^{n-\alpha-1} \\ = & \frac{1-t^2-Nt^4}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3} + C_1\frac{t^4+Nt^6}{(1-t^2-Nt^4)^{\frac{2}{n-\alpha-1}}} + Ct^{n-\alpha-1}.\end{aligned}$$

Now using the estimates for I_1 and I_2 we get

$$\begin{aligned}&\int_0^{b_N} t^{-q-1}\Big(A_{L,\xi,\alpha}(t)-A_{L,\xi,\alpha}(0)-A''_{L,\xi,\alpha}(0)\frac{t^2}{2}\Big)dt \\ \le & C_n\int_0^{b_N} t^{-q-1}\Big(\frac{1-t^2-Nt^4}{n-\alpha-1} - \frac{\alpha}{2}\frac{t^2}{n-\alpha-3} + C_1\frac{t^4+Nt^6}{(1-t^2-Nt^4)^{\frac{2}{n-\alpha-1}}} \\ & \qquad + Ct^{n-\alpha-1} - \frac{1}{n-\alpha-1} + \Big[\frac{\alpha}{n-\alpha-3}+\frac{2}{n-\alpha-1}\Big]\frac{t^2}{2}\Big)dt \\ = & C_n\int_0^{b_N} t^{-q-1}\Big(\frac{-Nt^4}{n-\alpha-1} + C_1\frac{t^4+Nt^6}{(1-t^2-Nt^4)^{\frac{2}{n-\alpha-1}}} + Ct^{n-\alpha-1}\Big)dt.\end{aligned}$$

Now one can estimate each term of the last integral separately. Since $b_N \simeq N^{-1/4}$, we get that

$$\int_0^{b_N} t^{-q-1}\frac{-Nt^4}{n-\alpha-1}dt \simeq -C_2N^{q/4}$$

for a positive constant C_2.

For the second term, we change the variable of integration: $u = N^{1/4}t$. Then

$$\begin{aligned}&\int_0^{b_N} t^{-q-1}\frac{t^4+Nt^6}{(1-t^2-Nt^4)^{\frac{2}{n-\alpha-1}}}dt \\ &\qquad = N^{q/4}\int_0^{b_NN^{1/4}} u^{-q-1}\frac{u^4N^{-1}+u^6N^{-1/2}}{(1-N^{-1/2}u^2-u^4)^{\frac{2}{n-\alpha-1}}}du \\ &\qquad \le N^{(q-2)/4}\int_0^{b_NN^{1/4}} u^{-q-1}\frac{u^4+u^6}{(1-N^{-1/2}u^2-u^4)^{\frac{2}{n-\alpha-1}}}du \\ &\qquad \le C_3N^{(q-2)/4},\end{aligned}$$

since $b_NN^{1/4} \to 1$ as $N \to \infty$, and the integral

$$\int_0^1 u^{-q-1}\frac{u^4+u^6}{(1-u^4)^{\frac{2}{n-\alpha-1}}}du$$

converges both at 0 and 1.

And finally, the integral of the last term is small for large values of N, since $n-\alpha-1=q+1$. From what we have obtained one can see that the integral over $[0,b_N]$ will be negative for large values of N since the leading term is $-C_2N^{q/4}$.

Now consider the integral over $[b_N,a_N]$. The expression

$$A_{L,\xi,\alpha}(t)-A_{L,\xi,\alpha}(0)-A''_{L,\xi,\alpha}(0)t^2/2$$

can be estimated from above by a constant C, not depending on N. Indeed, $A_{L,\xi,\alpha}(t) \le A_{L,\xi,\alpha}(0)$, $A''_{L,\xi,\alpha}(0)$ is a constant independent of N, and $t\le a_N \simeq N^{-1/4}\le 1$ for N large enough. Therefore

$$\int_{b_N}^{a_N} t^{-q-1}\Big(A_{L,\xi,\alpha}(t)-A_{L,\xi,\alpha}(0)-A''_{L,\xi,\alpha}(0)\frac{t^2}{2}\Big)dt$$
$$\le C\int_{b_N}^{a_N} t^{-q-1}dt \le C\int_{b_N}^{a_N}(b_N)^{-q-1}dt = C\frac{a_N-b_N}{(b_N)^{q+1}}.$$

Recalling that a_N and b_N satisfy the equations

$$1-a_N^2-Na_N^4=0 \quad\text{and}\quad 1-b_N^2-Nb_N^4=b_N^{q+1},$$

we conclude that

$$b_N^{q+1}=(a_N^2-b_N^2)(1+N(a_N^2+b_N^2)).$$

Therefore

$$C\int_{b_N}^{a_N} t^{-q-1}dt \le \frac{C}{(a_N+b_N)(1+N(a_N^2+b_N^2))} \simeq CN^{-1/4}.$$

Finally, the integral over $[a_N,\infty)$ can be computed as follows:

$$\int_{a_N}^{\infty} t^{-q-1}\Big(-A_{L,\xi,\alpha}(0)-A''_{L,\xi,\alpha}(0)\frac{t^2}{2}\Big)dt \simeq -D_1N^{q/4}+D_2N^{(q-2)/4}$$

where $D_1>0$. Therefore, this integral is negative for N large enough.

Combining all the integrals one can see that for N large enough the desired integral (8) is negative. This means that for some direction $\xi\in S^{n-1}$ the function $(\|x\|_L^{-1}\cdot|x|_2^{-\alpha})^\wedge(\xi)$ is negative, if $\alpha\in(n-5,n-4)$.

If $\alpha=n-5$, both sides of the equality in the statement of Lemma 2.2 vanish, therefore we need to apply the argument from [GKS] (see the proof of Theorem 1). Then

$$(\|x\|_L^{-1}\cdot|x|_2^{-n+5})^\wedge(\xi)=C\int_0^\infty t^{-4}\Big(A_{L,\xi,\alpha}(t)-A_{L,\xi,\alpha}(0)-A''_{L,\xi,\alpha}(0)\frac{t^2}{2}\Big)dt$$

for a positive constant C. Considering the same body as before, we get that $(\|x\|_L^{-1} \cdot |x|_2^{-n+5})^\wedge(\xi)$ is also negative at some point ξ. ∎

Now we are ready to finish the proof of the negative part. Apply Lemma 3.2 to construct an infinitely smooth origin-symmetric body L with positive curvature for which $(\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\xi) < 0$ for some direction ξ. By Lemma 2.2, the function $(\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge$ is continuous on the sphere S^{n-1}, hence there is a neighborhood of ξ where it is negative.

Let

$$\Omega = \{\theta \in S^{n-1}\colon (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta) < 0\}.$$

Choose a non-positive infinitely differentiable even function v supported on Ω. Extend v to a homogeneous function $r^{-\alpha-1}v(\theta)$ of degree $-\alpha-1$ on $\mathbb{R}^n$. By [K5, Lemma 3.16], the Fourier transform of $|x|_2^{-\alpha-1}v(x/|x|_2)$ is equal to $|x|_2^{-n+\alpha+1}g(x/|x|_2)$ for some infinitely differentiable function g on S^{n-1}.

Define a body K by

$$\|x\|_K^{-n+1} = \|x\|_L^{-n+1} + \varepsilon |x|_2^{-n+1} g(x/|x|_2)$$

for some small ε so that the body K is convex (see, for example, [K5, p. 96] for this standard perturbation argument). Multiply both sides by $\frac{1}{\pi(n-1)}|x|_2^\alpha$ and apply the Fourier transform:

$$(-\Delta)^{\alpha/2}S_K = (-\Delta)^{\alpha/2}S_L + \frac{\varepsilon(2\pi)^n}{\pi(n-1)}|x|_2^{-\alpha-1}v(x/|x|_2) \le (-\Delta)^{\alpha/2}S_L,$$

since v is non-positive.

On the other hand,

$$\begin{aligned}
&\int_{S^{n-1}} (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta)(-\Delta)^{\alpha/2}S_K d\theta \\
&= \int_{S^{n-1}} (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta)(-\Delta)^{\alpha/2}S_L d\theta \\
&\quad + \varepsilon \frac{(2\pi)^n}{\pi(n-1)} \int_{S^{n-1}} (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta) v(\theta) d\theta \\
&> \int_{S^{n-1}} (\|x\|_L^{-1} \cdot |x|_2^{-\alpha})^\wedge(\theta)(-\Delta)^{\alpha/2}S_L d\theta.
\end{aligned}$$

Repeating the argument from the proof of the affirmative part we get

$$\mathrm{vol}_n(L) < \mathrm{vol}_n(K). \qquad ∎$$

Remarks: (i) The negative part is formulated only for $q \in [n-5, n-4)$, because we wanted this to work for $n = 5$. In fact, for bigger n one can take $q \in [0, n-4)$. Also, the condition (1) can be written in terms of the Fourier transforms so that no smoothness of the bodies is required.

(ii) In the case where $q = n-4$ and n is an even integer, the result of Theorem 3.1 was proved in [K4] using an induction argument. The proof from [K4] cannot be extended to other values of q and n and does not produce any results in the negative direction.

(iii) Shephard's problem (see, for example, [K5, Section 8.4]) asks whether convex origin-symmetric bodies with smaller projections necessarily have smaller n-dimensional volume. As proved independently by Petty [P] and Schneider [S], the answer to this problem is affirmative only in dimension $n = 2$, so one may try to modify Shephard's problem to guarantee the affirmative answer in all dimensions. However, attempts to repeat the proof of Theorem 3.1 for Shephard's problem fail, since the section function $A_{K,\xi,p}$ may not be sufficiently differentiable.

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